METHOD OF CREATION FINITE-ELEMENTS TOPOLOGY MULTICOHERENT AREAS

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ABSTRACT
In paper, mathematical bases of a creation method of difficult configuration area representation in finite-elements are developed. The algorithm of task solution allows to unite simple areas and to order numberings of knots. Thus the sequence of knots points are formed, where tape width of nonzero coefficients of the whole equations system will be locally minimums.

INTRODUCTION
One of the main directions of scientific and technical researches is the creation of adequate numerical models of functioning of structural elements made of composite materials. Analysis of the strain state of structural elements of materials with non-uniform structure and geometry was a prerequisite for the development of numerical modeling using computer technology. In this regard, the urgent problem is creation a mathematically-based numerical models, computational algorithms and specialized software to create tools for the projection of new composite materials and calculation of construction elements. Widespread use in practice various composite materials promoted development of researches in the field of the anisotropic theory of plasticity. For the description of elastic-plastic deformation process of materials various versions of the theory of plasticity were offered. These are based on a method of averaging at using of which composite material will be replaced by homogeneous anisotropic environment.

Using of numerical models in three-dimensional tasks is important as analytical methods allow us to find solutions of private tasks, mainly for elementary areas. At present, an effective method of numerical analysis of constructions is the finite element method (FEM). FEM provides a solution to the three-dimensional elastic-plastic problems of deformation of composite structural elements of complex geometry. In this regard development of numerical model and computing algorithms of the solution for three-dimensional elastic-plastic tasks is actuals. Additively the creation of specialized software systems which will automate process of research and projection of new constructions made of composite materials is significant.

FEM is a projective and net method, in fact. However, at the initial stage it was developing as Bubnov-Galerkin-Ritts's method with a special choice of basic functions in the form of piecewise-determined polynoms on the grids of finite elements. This way of creation of FEM schemes is known as a variation - differential method. This interpretation of a method allowed extending estimates of approximation, speed of convergence and stability of variation methods to
computing FEM schemes. Extension of the area application has led to the development of the FEM as a projection method of the approximate solution of boundary value problems of mathematical physics.


Due to the lack of a single three-dimensional area decomposition algorithm for finite elements to build a discrete representation of the configuration of the field becomes complicated.

Formation and solution of high order equations system requires the development of special algorithms for data processing. It means that the solution of three-dimensional task and analysis of the results is a complex and time-consuming problem. In FEM the essential problem is to automate of creation of three-dimensional representation of area configuration. The articles H. A. Camel, G. K. Eisenstein, A. E. Prey, by C. A. Hall, T. A. Porsching, G. A. Keramidas, E. C. Ting, etc. are devoted to questions of creation of the grids for finite elements of three-dimensional areas.

The problem is solved by interactive processing of input and output data, as well as through developed specialized software. The technique of the solution of three-dimensional tasks and their program realization were considered in articles J. Argiris, Yu. Dankert, O. Zenkevich, L. Segerlind, R. Melosh, I. R. Rasheed, N. A. Vulfovich, A. P. Goryachev, V. V. Zarubayev, B. Kurmanbayev, V. A. Pakhomov, A. S. Sakharov, B. V. Fradkin, etc.

From the provided review we can conclude that for the solution of three-dimensional problems of physically nonlinear deformation of constructions made of composite materials important scientific and practical interest is represented by development of a technique, effective computing algorithms and the specialized software of creation of a finite-elements grid of the considered object.

When carrying out computing experiments on the basis of modeling of the solution of a problem of durability of constructions by of finite elements method it is necessary to automate process of creation of a finite-elements grid of the considered object [1]. If the area of a construction has a difficult configuration, creation of a finite-elements grid is the labor-intensive process demanding big ability and it is lot of time.
SOLUTION METHOD
The finite-elements grid of difficult object is formed by means of association of subareas of a simple configuration. Under idle time the area for which there is an algorithm of creation of a finite-elements grid is meant. We will enter the following definitions and criteria [2].

Definition 1.
Three-dimensional area in the form of system of the volume elements connected among themselves limited to the surfaces which are crossed in nodal points; boundary surfaces and lines, as well as each volume element, can have some number of internal knots; surfaces can be crossed only along boundary lines;

Two-dimensional area with system of the surfaces adjoining lengthways the boundary lines connected among themselves which are crossed in nodal points; thus boundary lines can also include some number of intermediate knots; surfaces can be limited to several lines; two lines have to connect so that one of them crossed another in a trailer point.

Definition 2.
Finite-elements representation of a configuration of area is described by a discrete set:

$$\Omega = \{ n, m, K, M \},$$

where

- $n$ – Number of knots of a finite-elements grid;
- $m$ – Quantity of finite elements;
- $K$ – Ordered set of coordinates of knots;
- $M$ – Ordered set of numbers of knots on finite elements.

Criterion 1. A condition of coincidence of boundary knots of two sets $\Omega_1$ and $\Omega_2$ is the ratio

$$|\delta_i - x_i^j| < \varepsilon \& |x_i^j - x_i^k| < \varepsilon \& |x_i^j - x_i^l| < \varepsilon,$$

Where

- $(x_i^1, x_i^2, x_i^3) \in K_1$ – set of coordinates of knots $\Omega_1$ ($i = 1, 2, \ldots, n_1$),
- $(x_j^1, x_j^2, x_j^3) \in K_2$ – set of coordinates of knots $\Omega_2$ ($j = 1, 2, \ldots, n_2$),
- $\varepsilon > 0$ – rather small number.

Theorem 1.
If at association of sets $\Omega_1 = \{ n_1, m_1, K_1, M_1 \}$ and $\Omega_2 = \{ n_2, m_2, K_2, M_2 \}$ satisfy conditions of topology of model of difficult area, then

$$m = m_1 + m_2,$$

$$n = n_1 + n_2 - q,$$
Where
\( m, n \) – total number of finite elements and knots,
\( q = |K_1 \cap K_2| \), – number of the knots located on border of association of subareas,
\( K_1 \) and \( K_2 \) - the ordered sets of coordinates of knots.

**Proof.**
As two subareas unite on boundary nodal points, total number of finite elements, is equal to their sum, i.e.
\[ m = m_1 + m_2. \]
According to definition 1, at association of two subareas which are crossed in boundary nodal points, knots coincide. In this case, total number of knots: \( n = n_1 + n_2 - q \), where \( q \) - number of coinciding boundary knots which is defined on criterion 1. Performance criterion 1 means, that border knots coincides to within rather small \( \varepsilon > 0. \) Size \( q \) will be equal to number of coinciding boundary knots. Thus, theorem 1 it is proved.

**Theorem 2.**
If at association of sets \( \Omega_1 = \{n_1, m_1, K_1, M_1\} \) and \( \Omega_2 = \{n_2, m_2, K_2, M_2\} \), satisfy conditions of topology of model of difficult area, then
\[ M = M_1 \cup M'_2, \quad K = K_1 \cup K'_2, \]
Where
\( K'_2 \) - the ordered set of coordinates of knots of a set \( \Omega_2 \), without boundary knots which coincide with boundary knots of a set of \( K_1 \subset \Omega_1 \);
\( M'_2 \) - the ordered set the renumbered knots of a set of \( M_2 \subset \Omega_2 \).

**Proof.**
According to the theorem 1, \( n = n_1 + n_2 - q \), \( m = m_1 + m_2 \).
For numbering of local numbers of knots of a set \( \Omega_1 \) and \( \Omega_2 \) are entered according to a set:
\[ N_1 = \{i \mid i \leq n_1\}, \quad i \in \mathbb{N} \quad \text{and} \quad N_2 = \{j \mid j \leq n_2\}, \quad j \in \mathbb{N}. \]
Further sets \( A \) and \( B \) are entered (\( |A| = |B| = q \)). Local numbers of coinciding boundary knots, respectively from sets of the \( N_1 \) and \( N_2 \) hubs satisfying to a ratio will be elements of these sets:
\[ A \times B = \left\{(i, j) \mid i \in N_1 \& \exists j \in N_2 : \sum_{k=1}^{l} \left| x'_i - x'_j \right| < l\varepsilon \right\}, \]
where
\( \varepsilon > 0 \) - to within rather small, \( l=3 \).

Formation of elements of a set of numbers of knots on finite elements \( M \) includes the following stages:

1) first \( k = 1,2,\ldots, m_1 \) are appropriated to elements of a set of \( M \) elements of a set of \( M_1 \), i.e. \( M_1 \subset M \);

2) the subsequent \( k = m_1+1, m_1+2,\ldots, m_1+m_2 \) elements of a set of \( M \) are formed by means of replacement of local numbers of knots of a set of \( M_2 \), on global numbers, i.e. \( M = M_1 \cup M'_2 \), where \( M'_2 \) - the set consisting of global numbers of knots.

Process of calculation of global numbers of knots is carried out by the following rule. If local numbers of a hub \( i \) of a set \( N_2 \) \( (i=1,2,\ldots, n_2) \), belongs to a set \( B \), i.e. \( i \in B \), is assigned to it the corresponding local number of knot from a set of \( A \). Otherwise, its value is calculated on the basis of a ratio: \( i' \rightarrow i + n_1 - z \). The size of a variable \( z \) is defined as quantity of elements of a set \( B \) which value there is less than a size of current issue of \( i \), i.e.

\[
z = |Q(\ i;\ B)|,\ \text{where } Q(\ i;\ B) = \{\ j \in B : j < i \}.
\]

Formation of elements of a set of coordinates of hubs \( K \) consists of the following:

1) first \( k = 1,2,\ldots, n_1 \) are appropriated to elements of a set of \( K \) elements of a set of \( K_1 \), i.e. \( K_1 \subset K \);

2) the subsequent \( k = n_1+1, n_1+2,\ldots, n_1+n_2 - q \) elements of a set of \( K \) are formed of \( K_2 \) subset elements, without coordinates of numbers of the knots which are in a set \( B \), i.e. \( K = K_1 \cup K'_2 \) where \( K'_2 \subset K_2 \) and \( |K'_2| = n_2 - q \).

Thus, all making sets \( \Omega \) – finite-elements representation of initial multicoherent area are defined. The theorem 2 is proved. Generalizing the above, we will prove the \textit{theorem 3}.

If for sets \( \Omega_i \) \( (i=1,2,\ldots,p) \), are satisfied conditions of topology of model of difficult area, for multicoherent area \( \Omega \) the ratio is carried out then,

\[
\Omega = \bigcup_{i=1}^{p} \Omega_i
\]

(2)

Where

\( p \) - Number of the subareas which are subject to association.

\textbf{Proof}.

At \( p=2 \), according to the theorem 1 and 2, the ratio (2) is carried out.
We will assume that (2) it is right at \( p-1 \), i.e.
\[ \Omega^{p-1} = \bigcup_{i=1}^{p-1} \Omega_i. \quad (3) \]

Then for multicoherent area \( \Omega \) we have:

\[ \Omega = \Omega^{p-1} \cup \Omega_p. \quad (4) \]

The ratio (4), according to the theorem 1 and 2, is also carried out. The theorem 3 is proved. Thus, the correctness of algorithm of creation of finite-elements representation of multicoherent area is proved.

The essence of streamlining consists in renumbering of knots on the basis of a frontal method [3]. In works [4] the algorithm of definition of the initial front is developed. The essence of algorithm is that knots of the initial front get out of the knots located on area border. Definitions of tops, boundary and internal knots are for this purpose entered. As initial fronts tops or set of boundary knots which are located between tops get out, including tops. On each initial front procedure of a frontal method [5] is carried out. On each finite element \( i \) the difference between maximum and minimum numbers of knots is calculated: \( h_i = N_{\max} - N_{\min} \). Then on a ratio \( l_k = \left( \max_{1 \leq i \leq N} h_i + 1 \right) \) the half width of a tape of the allowing system of the equations on this initial front, where by \( k \) – number of initial fronts is calculated. Local at least \( L \) width of a tape of nonzero coefficients of the allowing system of the equations which corresponds to the initial front, it is calculated as: \( L = \min_{1 \leq k \leq s} l_k \), where by \( s \) - number of initial fronts. As for any design the number of edges and side surfaces is limited, the number of searches of \( s \) is also limited [6].

**SOLUTION ALGORITHM**

For descriptive reasons we will consider process of formation of finite element model of two-dimensional area of a difficult configuration (figure 1.a).

![Figure 1. Initial area and forms of elementary subareas](image)
The following subareas are used as elementary areas (figure 1.b):

1) any quadrangle;
2) a rectangle with elliptic cut in top;
3) 1/4 part Torah;
4) 1/4 part of an ellipse.

Basic data for creation of a finite element grid of these areas are:

1) any quadrangle – coordinates of tops, number of splitting on axes of OX and OY;
2) a rectangle with elliptic cut in top – coordinates of the center and radiuses of an ellipse, the sizes of the parties of a rectangle, number of radial splitting’s and number of divisions on an axis OX;
3) 1/4 part a Torah - coordinates of the center and radiuses of an ellipse, number of radial splitting’s and number of divisions on an axis OX;
4) 1/4 part of an ellipse – coordinates of the center and radiuses of an ellipse, number of splitting’s on axes of coordinates.

Taking into account a configuration the studied area breaks into a set of elementary areas (figure 2.a)

![Figure 2. The studied area breaks into a set of elementary areas and formation of discrete model of a difficult configuration](image)

Formation of area of a difficult configuration through of using the library of elementary areas allows simplifying process of creation of finite element grid due to reduction of volume of the entered basic data. The algorithm of construction is reduced to a consecutive task of parameters of elementary subareas, formation of a finite element grid and their association (figure 2.b).

The described technique of creation of finite element model of area of a difficult configuration allows, without increasing quantity of finite elements and number of knots, to consider all geometrical features of a configuration of area.
ALGORITHM FOR THREE-DIMENSIONAL FINITE ELEMENT MODEL

Three-dimensional finite elements can be received by such operations as expression or leaving of a trace at rotation applied to the surfaces covered with a grid. We will consider as a design example in the form of a rectangular parallelepiped with ellipsoidal dredging in top (figure 3.a). It represents 1/8 part of a parallelepiped with an ellipsoidal cavity in the center.

Figure 3. A parallelepiped with ellipsoidal dredging in top and finite element representation of an ellipsoidal triangle

The beginning of system of coordinates we will arrange in the center of a cavity, and we will send to an axis of coordinates along edges. Rectangle sides with elliptic cut in a corner, break as follows. Ellipse points of intersection connect to axes of coordinates a straight line. The piece shares on n of equal parts. From the beginning of coordinates through points of splitting radial straight lines before crossing with an ellipse contour are drawn. Coordinates of points of intersection are defined from the following system of the equations:

\[
\begin{align*}
  x_i &= \frac{y}{y_i} \\
  x_i &= \frac{y}{y_i} \\
  x_i &= \frac{y}{y_i} \\
  \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{align*}
\]

Where \((x_i, y_i)\) - coordinates of points on a straight line \((i = 1, n)\),

n - Plural number,

a, b - ellipse half shafts.

Further, the average knot, received on an ellipse, connects to an opposite corner of a rectangle. The parties not adjacent to cut break into pieces in the same relation. The knots constructed on an ellipse connect to the knots received on the parties of a rectangle. These straight lines break into pieces on the basis of a proportion (5):

\[
\frac{a_i}{a} = \frac{l_i}{l} \quad i = 1, \ldots, m
\]

(5)
Where

- \(a_i\) – the set piece \(i\) length on the party of a rectangle, adjacent to cut,
- \(l\) – Length of the straight line connecting knot on an ellipse to the party of a rectangle,
- \(l_i\) – piece \(i\)-go length on this straight line,
- \(m\) – Quantity of pieces on the party of a rectangle, adjacent to an ellipse.

Length of a piece of a straight line is calculated on the following formula (6):

\[
l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},
\]

(6)

Where \((x_1, y_1)\) – knot coordinates on an ellipse, \((x_2, y_2)\) – knot coordinates on the party of a rectangle.

Coordinates of internal knots of a side are determined by ratios (7):

\[
\begin{align*}
x &= x_1 + \lambda_i x_2 \\
y &= y_1 + \lambda_i y_2
\end{align*}
\]

(7)

Where

\[
\lambda_i = \frac{\sum_{j=1}^{i} l_j}{\sum_{j=i+1}^{m} l_j}.
\]

Creation of discrete model of a surface of ellipsoidal dredging is connected with formation of a spatial triangle (figure 3.b) which angular knots have coordinates \((a;0;0),(0;b;0),(0;0;c)\) where \(a, b, c\) – ellipsoid half shafts.

The parties of this triangle share on \(n\) of equal parts. Then the geometrical center of a triangle connects to median knots of the parties of a triangle and the received pieces break as equal’s parts. In each of the received quadrangles we draw the straight lines connecting the splitting knots located on the opposite sides of quadrangles. Coordinates of knots on the parties of quadrangles are defined from the ratios similar (7), and coordinates of internal knots - from the decision of system of the equations:
Coordinates of the knots lying on the opposite sides of quadrangles. Coordinates of the knots located on a surface of ellipsoidal dredging decide from a condition of crossing of the radial straight lines passing through the knots lying on a triangle on an ellipsoid surface as follows:

\[
\begin{align*}
    x-x_i &= y-y_i = z-z_i \\
    x_2-x_i &= y_2-y_i = z_2-z_i \\
    x-x_3 &= y-y_3 = z-z_3 \\
    x_4-x_3 &= y_4-y_3 = z_4-z_3
\end{align*}
\]

Where
\[
(x_1, y_1, z_1)(x_2, y_2, z_2)(x_3, y_3, z_3)(x_4, y_4, z_4)
\]

Coordinates of the knots lying on the opposite sides of quadrangles. Coordinates of the knots located on a surface of ellipsoidal dredging decide from a condition of crossing of the radial straight lines passing through the knots lying on a triangle on an ellipsoid surface as follows:

\[
\begin{align*}
    x^2 + y^2 + z^2 &= 1 \\
    \frac{x}{a} = \frac{y}{b} = \frac{z}{c}
\end{align*}
\]

Where
\[
(x_i, y_i, z_i)
\]

ith coordinates — that knot, \((i = 1, \ldots, N)\),

\(N\) — Number of knots on a triangle, \(N=(n+1)(n/2+1)(n/2)^2\).

The final stage in creation of a finite element grid of a design is splitting other three sides which aren't adjoining dredging. Each side shares \(n^2/4\) on rectangles. Then the knots located on the middle of the parties of dredging and also in its tops connect to parallelepiped tops. To the central knot there has to correspond the point of intersection of three sides not adjacent to dredging. Splitting points on these straight lines are defined on the basis of proportions (5).

Thus, it is possible to hurt an initial body into hexagons. Numbering of knots of a body with ellipsoidal dredging in top is carried out by the frontal method described above. For what at first dredging knots are consistently numbered. Then, without interrupting a numbering order, knots of the subsequent layers are numbered. The number of knots on each layer will be identical. It should be noted that all layers in a form will be similar to dredging, except the last which is formed by three crossed parallelepiped sides. To form a final and element grid of a parallelepiped with a through cylindrical cavity in the middle, it is necessary to use final and element a rectangle grid with an elliptic cavity in top with addition of the coordinates set on \(OZ\) axis.

Coordinates of nodal points of each layer of a parallelepiped on axes of \(OX\) and \(OY\) coincide with coordinates of knots of a rectangle.

The algorithm of splitting a rectangle with elliptic dredging in top can be used for creation of finite-element representation of the hollow cylinder. For this purpose it is necessary to replace the equations of the parties of a rectangle with the equation

\[
And to add the corresponding values on OZ axis.

**ALGORITHM OF CONSTRUCTION**

The algorithm of construction includes the following stages:

1) formation of library of finite element models of elementary subareas;
2) procedure of association of subareas;
3) procedure of definition of the initial front of knots;
4) Procedure of streamlining of numbers of knots of finite element model.

Procedure of association of two subareas includes the following stages:

1) on the basis of comparison of coordinates of the knots located in the \( M_1 \) and \( M_2 \) are formed:
   - \( q \) - Number of the coinciding knots located on border of association of subareas;
   - The A and B including the corresponding numbers of knots located on border of association of subareas;
2) \( n = n_1 + n_2 - q; \)
3) \( m = m_1 + m_2; \)
4) initial \( M_1 \) to lines of the M array appropriates the corresponding values of the \( M'_1 \) sets \( \Omega_1 \);
5) The subsequent values of elements of lines of the M array are defined by means of procedure of replacement of local numbers of knots of a set \( \Omega_2 \), located in the \( M_2 \) array, on global numbers. If value of the current \( i \)-th numbers of the knot located in the \( N_2 \) is present at the \( M_2 \), the corresponding local number of knot from the A array is assigned to it. Otherwise, its value is calculated on the basis of a ratio: \( i' = i + n_1 - z \) where value of a variable \( z \) is defined as number of knots of the B array which number there is less than value \( i \);
6) for formation of initial \( n_1 \) of lines of the M array values of the \( M_1 \) array are used;
7) The next lines of the M array are formed of \( M'_2 \), with the withdrawn lines which numbers are specified in the \( M_2 \).

The final stage of algorithm of creation of finite element model is streamlining of numbers of knots that is connected with reduction of width of a tape of system of the allowing FEM equations. The essence of streamlining consists in renumbering of knots on the basis of a frontal method [3]. In the real work the frontal method is modified taking into account that the initial front gets out as sequence of numbers of the knots located on border of the considered area [4]. For streamlining of numbers of knots three fronts are used: in the first numbers of knots of initial or current fronts, in the second – numbers of knots previous settle down, and in the third – the new front is formed.

The algorithm of a method consists of the following stages:

1) As the initial front boundary knots get out;
2) Finite elements which contain knots with the same numbers, as well as numbers of knots of the front are defined;
3) Are excluded from (2) numbers of knots chosen on a step participating in current previous and in formed fronts;
4) the grid knots having identical numbers, as numbers of knots in the current front according to the following rule are renumbered: everyone the following numbered knot gets on unit a bigger number, than previous, and the initial renumbered knot has number one;
5) Contents of the current front (now it becomes previous) remain, contents of the new created front in flowing with its subsequent clarification are copied, i.e. the front "moves" on a finite-elements grid;
6) All actions described in points (2)-(5) until the created front becomes empty repeat.

On the example of two-dimensional area the best results turn out at a choice as the initial front of the ordered set of the knots located on edges and tops of a finite element grid of multicoherent area. In this regard we will enter the corresponding definitions.

In finite element representation of multicoherent area the knot which is found in the only finite element is called as top. The set of the knots located on border of area or on the border concluded between two tops is called as an edge. 
Process of search of an edge is carried out as follows:
1) in a random way the knot which is found in two finite elements gets out and is added to the front;
2) Search of the knots which are found only once in two finite elements described in point (1) is carried out. If those aren't present, search of the knots which are found two times if those aren't present is run in them, formation of the initial front comes to an end. Then it is cleared and carried out transition to point (1). Otherwise, number of this knot is added to the front;
3) If in the course of repeated use of the described actions the top is found in the second step two times, performance of search of an edge comes to an end and streamlining on the front is carried out;
4) All knots which are found one or two times, are added to the front and the actions described in points (2) - are carried out (3);
5) Steps (1) - repeat (4), all edges concluded between two tops won't be found yet.

To prevent it is necessary "to mark" and not to use the cycling of this process, all knots which are found in two final elements further. Other edges can be found replacement of a condition in the third point, i.e. the exit will come from a cycle if the "marked" knot is found.

COMPUTING EXPERIMENT
On the basis of this algorithm the software is developed and streamlining of numbers of knots of the two-dimensional multicoherent area (figure 4) consisting of association of triangular and quadrangular areas [4] is carried out.
The parties of a triangle are divided into $k$, and the parties of a quadrangle on $k/2$ of pieces, respectively. At $k=4$ reduction of a difference between maximum and minimum numbers of knots with 25 to 7 is observed. At $k=46$ – with 2209 to 49 (i.e. approximately by 45 times).

CONCLUSION
The developed way of formation of a finite element grid allows breaking into smaller elements of different vicinity of inclusions that fully captures the physical essence of process of deformation of the bodies subject to external influences. The constructive elements considered above having cavities or inclusions can be components of the areas representing constructional composite materials, conglomerates of disperse particles in material, deposits of rocks, etc.

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